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COMAP

Module 327

Nuclear Deterrence

Harvey A. Smith

"After the second world war, the United States' monopoly of nuclear weapons permitted it to attempt to deter aggression by announcing a policy of "massive retaliation" against any nation which might attack the U.S. or its allies. As the Soviet nuclear capability expanded during the 1950's, the effectiveness of this policy as a deterrent came into question."

Applications of Elementary Probability to International Relations

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NUCLEAR DETERRENCE

by

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Intermodular Description Sheet: *UMAP Unit 327*

Title: NUCLEAR DETERRENCE

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Suggested Support Materials:

References: See Section 5 of text.

Prerequisite Skills:
1. Compute probabilities using the binomial distribution.
2. Perform elementary algebraic operations.
3. Compute roots of quadratic equations.
4. Follow a mathematical argument.

Output Skills:
1. Apply mathematical models to nuclear deterrence problems.
2. Express enlightened skepticism about the use of linear models in stability analyses and long-term predictions.

Other Related Units:
The Richardson Arms Race Model *(Unit 308)*
The Geometry of the Arms Race *(Unit 311)*

NUCLEAR DETERRENCE

INTRODUCTION

After the second world war, the United States' monopoly of nuclear weapons permitted it to attempt to deter aggression by announcing a policy of "massive retaliation" against any nation which might attack the U. S. or its allies. As the Soviet nuclear capability expanded during the 1950's, the effectiveness of this policy as a deterrent came into question. It was pointed out (notably by A. Wohlstetter [1]) that a well-coordinated surprise nuclear attack upon U.S. airbases could conceivably reduce the retaliatory force to a point where the residual retaliation might be risked by a determined or desperate aggressor.

This development prompted emphasis on maintaining forces which could withstand an all-out Soviet attack and still retain the ability to visit overwhelming retaliation upon the aggressor's population centers. The retaliatory attack was to be sufficient to assure the destruction of the aggressor state as a viable society [2], so the policy became known as an *assured destruction* policy.

To assure partial invulnerability, part of the manned bomber force was kept airborne at all times and another part was maintained in a state of readiness to take off on short notice. The bomber force was supplemented by missiles carried aboard nuclear submarines and by land-based missiles emplaced in underground reinforced concrete "silos". Each of these three forces was to be able to carry alone the burden of deterrence should a surprise technological development render the other two vulnerable to attack.

The effectiveness of a retaliatory force as a deterrent depends, of course, on how much of it can be destroyed by a surprise attack and hence on the forces a potential aggressor can commit to such an attack. During the past two decades Soviet nuclear forces have increased rapidly. At least two explanations can be advanced for this increase. It might be that the Soviet leaders are seeking the ability to reduce the U.S. deterrent force so greatly by a surprise attack that they would feel they could survive the subsequent retaliation. (This is not to say they intend to make a surprise attack; the *ability* to do so could suffice to gain concessions.) On the other hand, it might be that they merely wish to have the same sort of deterrent against the U.S. which the U.S. holds against them. In the former case, if the U.S. attempts to maintain its deterrent what are the relative economic burdens on the two parties in the resulting arms race? In the latter case, does the mutual attempt at deterrence lead to an unlimited arms race or does it settle down to an equilibrium in which each party is satisfied that the other is effectively deterred? In recent years, other nations than the U.S. and the Soviet Union have begun to develop nuclear weapons. These weapons may eventually come to play a significant role in the strategic calculations of the great powers. What will be their effect? These are the questions we address in this module. If we pose the problem in complete generality, allowing for all types of forces and the possibility of each type attacking or being attacked by all the others, the formulation becomes overwhelmingly complex. We will avoid this by concentrating solely on land based missile forces. Because of the policy that each of the forces should be capable of serving alone as a deterrent [2] and because of the current relative inefficiency of attacking land based missles from bombers and submarines, this restricted problem is more relevant than it might at first seem, even though it does not account for the full range of interactions which must be considered in force planning.

1. A LINEAR MODEL

In casual talk one might say something like, "Suppose it takes two Soviet missiles, on the average, to destroy one U.S. missile." The first model we present is based on a general "exchange ratio" concept of this type.

Suppose n different parties are attempting to maintain mutual deterrence. It is presumed that alliance between these parties may occur in any fashion. (A dependable long-term alliance can be regarded as a single party.) Each party must be prepared to deter not only each one of the others individually but the most threatening possible alliance of all other parties against it. Suppose party i has M_i missiles and that it takes ρ_{ji} of party j's missiles, "on the average" to destroy one of party i's missiles. Then the fraction

$$(1) \qquad \frac{M_j}{\rho_{ji}} = \frac{\text{The Number of Missiles j has}}{\text{The Number of Missiles j needs to Destroy One of i's Missiles}}$$

is the number of i's missiles that j is likely to be able to destroy. Accordingly, the sum

$$(2) \qquad \sum_{j \neq i} M_j/\rho_{ji}$$

is the number of missiles that i would be likely to lose in a coordinated attack by all the other parties.

Suppose also that party i judges that the expectation of his having Γ_i missiles operational for retaliation after attack would suffice to deter such an attack. Then party i will feel secure if

(3) $$M_i - \sum_{j \neq i} M_i/\rho_{ji} \geq \Gamma_i .$$

If party i wants to minimize the number of missiles (and hence, presumably, costs) the equality will hold. We can thus find the minimal mutual deterrence posture by solving the system of simultaneous equations

(4) $$M_i - \sum_{j \neq i} M_j/\rho_{ji} = \Gamma_i, \qquad i = 1,2,\ldots,n.$$

Note, however, that this will be meaningful only if all the solutions are positive. If some solution M_i turns out to be negative, the conditions (3) cannot be met in reality and attempting to achieve mutual deterrence on these terms will result in an unlimited arms race.

It is instructive to write out specific solutions for the two party case (n = 2). We have from (4)

(5) $$\begin{aligned} M_1 - M_2/\rho_{21} &= \Gamma_1 \\ -M_1/\rho_{12} + M_2 &= \Gamma_2 . \end{aligned}$$

Solving this system yields

(6) $$M_1 = \frac{\rho_{12}\Gamma_2 + \rho_{12}\rho_{21}\Gamma_1}{\rho_{12}\rho_{21} - 1}$$

$$M_2 = \frac{\rho_{21}\Gamma_1 + \rho_{21}\rho_{12}\Gamma_2}{\rho_{21}\rho_{12} - 1}.$$

Thus in this model stable mutual deterrence can exist provided $\rho_{12}\rho_{21} > 1$; in particular if it takes at least one missile, "on the average", to destroy a missile. This might seem a reasonable condition for missiles carrying one warhead. (Silos are spaced far enough apart to avoid the possibility of more than one being damaged by a single warhead.) The introduction of multiple independent warheads on each missile (MIRV), however, makes the assumption questionable since the independent warheads from a single missile can attack several silos.

Some further insight can be gained by considering the totally symmetric case in which each party has the same technology and desires so that $\rho_{ij} = \rho$, $\Gamma_i = \Gamma$, $M_i = M$ for all i and j. We then have from (4)

$$M - (n - 1) M/\rho = \Gamma$$

and hence

$$M = \rho\Gamma/(\rho + 1 - n) \ . \tag{7}$$

Thus we see that in this case there can be no stable mutual deterrence unless $\rho + 1 > n$. For the three party case, stability cannot exist unless $\rho > 2$, i.e. it takes more than two missiles to destroy a single missile. Since the technology available in the early 1970's was widely believed to yield $\rho < 2$, we would conclude from this model that an attempt by three equal parties to achieve mutual deterrence using current technology would lead to an infinite arms race. Thus the development of MIRV and the development of many nuclear powers appear in this model to threaten even a theoretical possibility of stability. We will see in the next section that although these developments may indeed be escalatory, their effect is vastly overestimated by the model we have presented in this section.

Exercises

1.1 Verify that the non-symmetric two party solution yields the symmetric solution in case $\rho_{12} = \rho_{21}$, $\Gamma_1 = \Gamma_2$.

1.2 In a two-party situation, suppose only one party is following a deterrent policy. The other party is building missiles at a rate $M_2(t) = kt$. Find $M_1(t)$, the minimal deterrent force the first party must hold as a function of time, assuming $M_1(0) = \Gamma_1$. Draw a graph of $M_1(t)$ if $k = 100$ missiles per year, $\rho_{12} = \rho_{21} = 2$ and $\Gamma_1 = 200$.

1.3 For the symmetric case it is not unreasonable to presume that the required retaliatory force is proportional to the number of parties to be deterred; $\Gamma = (n - 1)\gamma$. Assuming this and $\rho = 4$, compute M/γ as a function of n for n = 2, 3, 4. (Note that it becomes infinite for n = 5.)

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2. A NONLINEAR MODEL

The model presented above embodies a fallacy in its very formulation. The concept of an exchange ratio "on the average", although it appears acceptable in verbal "analysis", turns out to be technically meaningless unless we specify the size of forces in advance. It is fallacious to treat as a "constant" a quantity which depends on the sizes of the forces when we attempt to compute the required force sizes. A more careful and exact probabilistic analysis is required. Let us suppose that each of the M_j missiles of party j carries μ_j independently targeted warheads, each having probability p_{ij} of destroying one of party i's M_i missiles in its silo if it is targeted on it. If M_i divides $\mu_j M_j$ evenly, then each of M_i missiles would be targeted by $\mu_j M_j / M_i$ of j's warheads. If this division is not even, the leftover warheads will be distributed over silos chosen at random from among the M_i. (It can be proved that this is the most effective attack, but the mathematics is more complicated than we require for this module. The theorem needed is given in [4].) The probability of a particular missile being chosen for the residual targeting is thus just the fractional part of $\mu_j M_j / M_i$. For convenience, let us introduce the notation $[x]$ for the integer part of x (the greatest integer $\leq x$) and $\langle x \rangle$ for the fractional part of x, $\langle x \rangle = x - [x]$. Then each of the M_i missiles is definitely targeted by $[\mu_j M_j / M_i]$ warheads and the probability of a particular missile being targeted and destroyed by one of the leftover warheads is $\langle \mu_j M_j / M_i \rangle p_{ij}$. Each warhead's attack on a missile can be regarded as an independent trial, so the probability of a particular missile surviving the attack by party j's missiles is

$$(1 - p_{ij})^{[\mu_j M_j / M_i]} (1 - \langle \mu_j M_j / M_i \rangle p_{ij}) .$$

Thus the condition that the expected number of missiles surviving attack by a coalition of all other parties should exceed Γ_i is

$$(8) \qquad M_i \prod_{j \neq i} (1 - p_{ij})^{[\mu_j M_j / M_i]} (1 - \langle \mu_j M_j / M_i \rangle p_{ij}) \geq \Gamma_i .$$

Two conclusions are immediately apparent. First, stable mutual deterrence postures always exist. Second, no particular parity or ratio between the missiles held by the various parties is necessary. Mutual deterrence postures exist for *any* specified ratio. To see these results, consider any arbitrary collection replaced by their positive multiples $\alpha M_1, \alpha M_2, \alpha M_3, \ldots, \alpha M_n$, the terms $\mu_j M_j / M_i$ appearing in (8) remain unchanged, so the left side of the inequality is increased by precisely the factor α. Thus, by choosing α sufficiently large, the forces $\alpha M_1, \alpha M_2, \alpha M_3, \ldots, \alpha M_n$ will constitute mutual deterrence postures for all the parties.

The problem of solving the minimal deterrence problem obtained by setting the inequalities (8) to equality is quite complicated and algorithms for its solution are discussed in detail in [3]. Here we will only consider the symmetric case, which is easily solved and which is quite indicative of the general behavior of the solutions.

Let $M_i = M$, $p_{ij} = p$, $\mu_j = \mu$. We might also postulate that the retaliatory force required is proportional to the number of parties to be deterred:

$$\Gamma_i = (n-1)\gamma.$$

For this symmetric case (8) becomes

$$M(1 - p)^{(n-1)\mu} \geq (n - 1)\gamma \tag{9}$$

so minimal deterrence is achieved by

$$M = (n - 1)\gamma(1 - p)^{-(n-1)\mu}. \tag{10}$$

If we regard M as a function of n we can write this as

$$\frac{M(n)}{\gamma} = (n - 1)\left[\frac{M(2)}{\gamma}\right]^{n-1} . \tag{11}$$

Suppose for instance that in deterrence between two equal parties each requires M(2) = 1000 missiles in order that a retaliatory force $\gamma = 200$ should survive an all-out attack by the other, then $M(2)/\gamma = 5$. The introduction of a third equal party, with each attempting to deter the other two, would require

$$\frac{M(3)}{200} = 2\cdot(5)^2 = 50 ,$$

so each of the three parties would require 10,000 missiles rather than the 1000 required for the two party case. For four parties we have

$$\frac{M(4)}{200} = 3\cdot(5)^3 = 375$$

so each party would need 75,000 missiles. We see that even though it is not theoretically impossible for more than two parties to establish mutual deterrence, the demands on resources posed by such a task are excessive and may be practically impossible. It is interesting to speculate that this practical difficulty may necessitate formation of the sort of stable long-term alliances which we noted earlier could be treated as a single party for planning purposes.

An alternative situation of interest is that in which a smaller third power acquires a relatively small nuclear capability consisting of M_3 missiles without attempting to hold a deterrent posture against the two major powers. This third power could collude with either of the two major powers in an attack on the other and it might attempt to wrest some advantage from this ability to influence the "balance of power." By how much must the major powers increase their forces to neutralize this effect? Again we assume equality of the major powers; $\mu_3 M_3 < M = M_1 = M_2$ and write p_3 for $p_{13} = p_{23}$. The mutual deterrence equations for parties 1 and 2 against the possible alliance becomes

$$(12) \qquad M(1-p)^{\mu}(1-p_3)^{[\mu_3 M_3/M]}(1-p_3\langle\mu_3 M_3/M\rangle) = \Gamma.$$

Since we are presuming $\mu_3 M_3 < M$, we have $[\mu_3 M_3/M] = 0$, $\langle\mu_3 M_3/M\rangle = \mu_3 M_3/M$ and hence

$$(13) \qquad M = \Gamma(1-p)^{-\mu} + p_3\mu_3 M_3 .$$

Thus it suffices for each of the major powers merely to increase its missile forces by an amount somewhat less than the total number of warheads mounted by the third power in order to neutralize any influence the third power may attempt to exercise.

Exercises

2.1 For the symmetric case $\mu_1 = \mu_2 = 2$, $\Gamma_1 = \Gamma_2 = 200$, $p_{12} = p_{21} = \frac{1}{2}$, suppose that party 2 insists on always having twice as many missiles as party 1: $M_2 = 2M_1$. Find the minimal deterrence posture satisfying this additional constraint.

2.2 Verify that for the two party case $p_{12} = p_{21} = \frac{1}{2}$, $\mu_1 = \mu_2 = 1$, $\Gamma_1 = 100$, $\Gamma_2 = 150$, the least cost solution is $M_1 = 220$, $M_2 = 260$.

2.3 In a two party situation, suppose that only one party is following a deterrent policy. The other party is building missiles at a constant rate $M_2(t) = kt$. Draw a graph of $M_1(t)$ if $k = 100$ missiles per year, $p_{12} = p_{21} = \frac{1}{2}$, $\mu_1 = \mu_2 = 1$, $\Gamma_1 = 200$, assuming $M_1(0) = \Gamma_1$.

2.4 Assuming $p = 1/8$, $\mu = 2$, compute $M(n)/\gamma$ for $n = 2, 3, 4, 5$ for the symmetric n-party situation.

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3. RELATIONSHIP BETWEEN THE TWO MODELS

At first glance, there would appear to be little relationship between the two models we have discussed. In fact we will show that the linear model is an approximation to the nonlinear model which holds when the probabilities p_{ij} are very small, i.e. when the missiles are virtually invulnerable to one another. Thus the linear model was not an inappropriate one to use in the late 1950's and early 1960's when this was generally believed to be the case.

Suppose x is a very small quantity. By the binomial theorem we know that

$$(14) \qquad (1 - x)^m = 1 - mx + \frac{m(m-1)}{2} x^2 + \ldots + (-x)^m .$$

If x is very small, its higher powers are even smaller. (If $x = 0.1$, $x^2 = 0.01$, $x^3 = 0.001$.) Thus terms involving these higher powers can be neglected in the expression above and we can write approximately

$$(15) \qquad (1 - x)^m \approx 1 - mx \quad .$$

Let us apply this observation to the nonlinear deterrence inequality (8) assuming p_{ij} to be so small that terms involving its higher powers can be neglected. That is, let us use (15) with $x = p_{ij}$ and $m = [\mu_j M_j/M_i]$ to replace the expression

$$(1 - p_{ij})^{[\mu_j M_j/M_i]}$$

in (8) by the expression

$$1 - [\mu_j M_j/M_i]p_{ij}$$

When this is done, (8) becomes

$$(16) \qquad M_i \prod_{j \neq i} (1 - [\mu_j M_j/M_i]p_{ij})(1 - \langle \mu_j M_j/M_i \rangle p_{ij}) \geq \Gamma_i .$$

In computing the product we may further eliminate any term involving more than one factor p_{ij}, since such terms will also be negligible. Eliminating such terms, we obtain

$$(17) \qquad M_i \left(1 - \sum_{j \neq i} [\mu_j M_j/M_i] p_{ij} - \sum_{j \neq i} \langle \mu_j M_j/M_i \rangle p_{ij} \right) \geq \Gamma_i .$$

Since $[x] + \langle x \rangle = x$, this is

$$(18) \qquad \begin{aligned} &M_i \left(1 - \sum_{j \neq i} (\mu_j M_j/M_i) p_{ij} \geq \Gamma_i \right) \\ &M_i - \sum_{j \neq i} \mu_j M_j p_{ij} \geq \Gamma_i \end{aligned}$$

which has precisely the form of (3) if the exchange ratios ρ_{ji} are taken to be $1/\mu_j p_{ij}$, i.e, each of j's missiles destroys "on the average" $\mu_j p_{ij}$ of i's, which is quite in accord with our expectations about what these ratios should be. The linear model is thus a fairly good approximation in restricted circumstances. The difficulties arise when one attempts to apply the model outside these circumstances. Then it can be quite misleading, and one must resort to the more complicated nonlinear model to gain insight into the problem. In applying "reasonable seeming" linear models of any process, one must always be wary of drawing conclusions which involve quantities becoming very large. (Questions of stability usually fall in this category.) Almost invariably the linear model must be replaced by a more complicated nonlinear model to deal successfully with such questions.

Exercises

3.1 Complete the solution given by the linear model to the two party case for $\rho_{21} = \rho_{12} = 2$, $\Gamma_1 = 100$, $\Gamma_2 = 150$ and compare it with the solutions to the nonlinear model verified in Exercise 2.2. What do you conclude?

3.2 Compare the results of Exercise 1.2 and 2.3. What do you conclude?

3.3 Compare the results of Exercise 1.3 and 2.4. What do you conclude?

---

4. ON THE USE OF AVERAGE VALUES

In all of the computations above we have dealt with mean values. The criterion for deterrence was that the *expected* retaliatory force should be adequate. The number of missiles actually surviving attack would be a random variable. There remains, therefore, the question of how much confidence can be placed in the

If x is very small, its higher powers are even smaller. (If $x = 0.1$, $x^2 = 0.01$, $x^3 = 0.001$.) Thus terms involving these higher powers can be neglected in the expression above and we can write approximately

$$(15) \qquad (1 - x)^m \approx 1 - mx \quad .$$

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When this is done, (8) becomes

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In computing the product we may further eliminate any term involving more than one factor p_{ij}, since such terms will also be negligible. Eliminating such terms, we obtain

$$(17) \qquad M_i \left(1 - \sum_{j\neq i}[\mu_j M_j/M_i]p_{ij} - \sum_{j\neq i}\langle\mu_j M_j/M_i\rangle p_{ij}\right) \geq \Gamma_i .$$

Since $[x] + \langle x\rangle = x$, this is

$$M_i\left(1 - \sum_{j\neq i} (\mu_j M_j/M_i)p_{ij} \geq \Gamma_i\right)$$

(18)

$$M_i - \sum_{j\neq i} \mu_j M_j p_{ij} \geq \Gamma_i$$

which has precisely the form of (3) if the exchange ratios ρ_{ji} are taken to be $1/\mu_j p_{ij}$, i.e, each of j's missiles destroys "on the average" $\mu_j p_{ij}$ of i's, which is quite in accord with our expectations about what these ratios should be. The linear model is thus a fairly good approximation in restricted circumstances. The difficulties arise when one attempts to apply the model outside these circumstances. Then it can be quite misleading, and one must resort to the more complicated nonlinear model to gain insight into the problem. In applying "reasonable seeming" linear models of any process, one must always be wary of drawing conclusions which involve quantities becoming very large. (Questions of stability usually fall in this category.) Almost invariably the linear model must be replaced by a more complicated nonlinear model to deal successfully with such questions.

Exercises

3.1 Complete the solution given by the linear model to the two party case for $\rho_{21} = \rho_{12} = 2$, $\Gamma_1 = 100$, $\Gamma_2 = 150$ and compare it with the solutions to the nonlinear model verified in Exercise 2.2. What do you conclude?

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3.3 Compare the results of Exercise 1.3 and 2.4. What do you conclude?

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4. ON THE USE OF AVERAGE VALUES

In all of the computations above we have dealt with mean values. The criterion for deterrence was that the *expected* retaliatory force should be adequate. The number of missiles actually surviving attack would be a random variable. There remains, therefore, the question of how much confidence can be placed in the results. Perhaps an average value is misleading and one must have greater forces to have sufficient confidence that the required retaliatory forces would survive. One can compute, for any case, the increase in force size required to provide any desired level of confidence in having the specified retaliatory force survive. In practical cases the force increase required for 95% confidence has always been found to be substantially less than 20% of the forces indicated by the mean calculations. The increase for 80% confidence is usually substantially below 10%. Thus the use of mean values in our calculations is not grossly misleading and only a moderate upward adjustment of force sizes is required if the parties demand high confidence in the survival of an adequate deterrent. To see how such calculations are done, let us consider the case of two equal parties seeking a minimal mutual deterrent. We have seen that the probability of a missile surviving an all-out attack in this case is $P = (1 - p)^{\mu}$. The probability of exactly N out of M surviving is given by the binomial distribution.

$$(19) \qquad \frac{M!}{N!(M-N)!} P^N (1 - P)^{M-N}$$

and the probability of at least Γ surviving is

$$(20) \qquad \sum_{N=\Gamma}^{M} \frac{M!}{N!(M-N)!} P^N (1-P)^{M-N} \quad .$$

This could easily be computed for small M, but for $MP(1 - P)$ large, it is more convenient to use the normal approximation to the binomial distribution and consult tables of the error function.

Exercises

4.1 If $MP(1 - P)$ is much greater than 1 then the sum

$$\sum_{N=\Gamma}^{M} \frac{M!}{N!(M-N)!} P^N (1-P)^{M-N}$$

is closely approximated by

$$\frac{1}{\sqrt{2\pi}} \int_K^{\infty} e^{-x^2/2} \, dx = E(K)$$

where $K = (\Gamma - MP - 1/2)/\sqrt{MP(1-P)}$.

(The error in this approximation is less than 0.03 for $P = 1/8$ and $M > 1000$.)

Given that

$E(-0.84) = 0.8$

$E(-1.28) = 0.9$

$E(-1.65) = 0.95$,

compute the excess number of missiles (over that indicated by the mean value computation) required respectively for 0.8, 0.9, and 0.95 confidence in having a retaliatory force of $\Gamma = 250$ survive attack in a two-party symmetric case where

a) $p = 1/2$, $\mu = 3$

b) $p = 3/4$, $\mu = 3$.

(The nonlinear model is to be used, of course.)

---

5. REFERENCES

[1] Wohlstetter, A., "The Delicate Balance of Power," Foreign Affairs, January 1959, pp. 211-234.

[2] Hearings, Department of Defense Appropriations, Part I, Senate Committee on Appropriations and Senate Committee on Armed Services, pp. 43-44, 1965.

[3] Kupperman, R.H. and Smith, H.A., "Some Computational Techniques and Theorems Concerning Mutual

Deterrence Relationships," Journal of Cybernetics, Vol. 1, No. 4, pp. 4-31, (1971).

[4] Kupperman, R. H. and Smith, H.A., The Role of Population Defense in Mutual Deterrence, Appendix A., A general theorem on the allocation of resources, SIAM Review, Vol. 19, No. 2. April 1977. pp. 306-308.

6. ANSWERS TO EXERCISES

1.1 $M = \frac{\rho\Gamma + \rho^2\Gamma}{\rho^2 - 1} = \frac{\rho(\rho + 1)\Gamma}{(\rho + 1)(\rho - 1)} = \frac{\rho\Gamma}{\rho - 1}.$

1.2 $M_1 = \Gamma_1 + M_2/\rho_{21} = \Gamma_1 + kt/\rho_{21}$

$= 200 + 50t$

$M_2 = 100t.$

1.3 $\frac{M}{\gamma} = \frac{\rho(n - 1)}{\rho + 1 - n} = \frac{4(n - 1)}{5 - n}$

$n = 2, \frac{M}{\gamma} = \frac{4}{3} = 1.33$

$n = 3, \frac{M}{\gamma} = \frac{8}{2} = 4$

$n = 4, \frac{M}{\gamma} = \frac{12}{1} = 12$

$n = 5, \frac{M}{\gamma} = \infty$ (instability).

2.1 $$M_1(1 - p)^{\left[\mu \frac{M_2}{M_1}\right]}(1 - p<\mu\frac{M_2}{M_1}>) \geq \Gamma$$

$$M_2(1 - p)^{\left[\mu \frac{M_1}{M_2}\right]}(1 - p<\mu\frac{M_1}{M_2}>) \geq \Gamma.$$

If $M_2 = 2M_1$, $\mu = 2$, $p = 1/2$, $\Gamma = 200$ this becomes

$$M_1(1 - \tfrac{1}{2})^4 \geq 200$$

$$2M_1(1 - \tfrac{1}{2}) \geq 200$$

$$M_1 \geq 3200, \; M_1 \geq 200.$$

Thus the solution is $M_1 = 3200$, $M_2 = 6400$.

2.2 $$\left[\mu\frac{M_2}{M_1}\right] = \left[\frac{260}{220}\right] = 1, \quad <\mu\frac{M_2}{M_1}> = \frac{40}{220} = \frac{2}{11}$$

$$\left[\mu\frac{M_2}{M_1}\right] = \left[\frac{220}{260}\right] = 0, \quad <\mu\frac{M_1}{M_2}> = \frac{220}{260} = \frac{11}{13}$$

$$M_1(1 - p)^{\left[\mu\frac{M_2}{M_1}\right]}\left(1 - p<\frac{M_2}{M_1}>\right) = \Gamma_1$$

$$220 \cdot \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2} \cdot \frac{2}{11}\right) = 110 \cdot \frac{10}{11} = 100$$

$$M_2(1 - p)^{\left[\mu\frac{M_1}{M_2}\right]}\left(1 - p<\mu\frac{M_1}{M_2}>\right) = \Gamma_2$$

$$260 \cdot \left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2} \cdot \frac{11}{13}\right) = 260 \cdot \frac{15}{26} = 150.$$

2.3 $$M_1(1 - p_{12})^{\left[\mu \frac{M_2}{M_1}\right]} \left(1 - p_{12} \langle \mu \frac{M_2}{M_1} \rangle\right) = \Gamma_1$$

$$M_2 = kt$$

$$M_1(1 - p_{12})^{\left[\mu \frac{M_2}{M_1}\right]} \left(1 - p_{12} \langle \mu \frac{kt}{M_1} \rangle\right) = \Gamma_1.$$

Let $n = [100t/M_1]$.

$$M_1\left(1 - \frac{1}{2}\left(\frac{100t}{M_1} - n\right)\right) = 2^n \cdot 200$$

$$M_1(1 + \frac{n}{2}) = 2^n \cdot 200 + 50t.$$

Initially, $n = 0$ so

$$M_1 = 200 + 50t$$

until $100t = M_1 = 200 + 50t$ (i.e., at $t = 4$), when $n = 1$, then

$$M_1 = \frac{800 + 100t}{3}$$

until $100t = 2M_1 = \frac{1600 + 200t}{3}$ (i.e., at $t = 16$), when $n = 2$, then

$$M_1 = 400 + 25t$$

and so forth.

2.4 $$\frac{M}{\gamma} = (n - 1)(1 - p)^{-(n-1)\mu}$$

$$= (n - 1)(\frac{7}{8})^{-\mu(n-1)} .$$

$$= (n - 1)(\frac{64}{49})^{n-1} .$$

$$\frac{M(2)}{\gamma} = 1.306$$

$$\frac{M(3)}{\gamma} = 3.412$$

$$\frac{M(4)}{\gamma} = 6.685$$

$$\frac{M(5)}{\gamma} = 11.641.$$

3.1 The linear model yields

$$M_1 = \frac{2 \cdot 150 + 4 \cdot 100}{4 - 1} = \frac{700}{3} = 233\ 1/3$$

$$M_2 = \frac{2 \cdot 100 + 4 \cdot 150}{3} = \frac{800}{3} = 266\ 2/3.$$

The nonlinear model yielded $M_1 = 220$, $M_2 = 260$. The linear model gives a fair approximation for p small and small total numbers of weapons, but it overestimates the requirement for weapons.

3.2 The two models agree out to $t = 4$, where $M_1 = M_2 = 400$ Beyond that point the linear model overestimates the required M_1. The overestimation progressively increases as M_2 increases and eventually becomes vast. According to the linear model, this arms race costs the would-be agressor twice as much annually as the defender. According to the non-linear model, the defender has an ever increasing advantage. The cost ratio becomes 3:1 after 4 years, 4:1 after 16 years, etc.

3.3 The linear model again overestimates the required forces in all cases. The error is not bad for $n = 2$, but it becomes progressively worse as the number of parties increases.

4.1 $\Gamma^2 + M^2P^2 + \frac{1}{4} - 2\Gamma MP - \Gamma + MP = MP(1 - P)K^2$

$$P^2M^2 + \left(P - 2\Gamma P - P(1 - P)K^2\right)M + \Gamma^2 - \Gamma + \frac{1}{4} = 0$$

$$M = \frac{\Gamma}{P} + \frac{K^2(1-P)}{2P}\left[1 + \sqrt{1 + \frac{4\Gamma - 2}{K^2(1-P)}}\right] - \frac{1}{2P}.$$

The nominal value of M given by the nonlinear model is $\Gamma/(1-p)^{\mu} = \frac{\Gamma}{P}$. Thus the excess required to provide confidence is

$$\Delta M = \frac{K^2(1-P)}{2P}\left[1 + \sqrt{1 + \frac{4\Gamma - 2}{K^2(1-P)}}\right] - \frac{1}{2P}.$$

For 4.1a we have $p = (\frac{1}{2})$, $\mu = 3$, $P = (1-p)^{\mu} = \frac{1}{8}$, $\Gamma = 250$. Thus

$$\Delta M = K^2 \cdot \frac{7}{2}\left[1 + \sqrt{1 + \frac{998 \cdot 8}{7K^2}}\right] - 4$$

$$= 3.5\left(K^2 + \sqrt{K^4 + 1141K^2}\right) - 4.$$

For 0.8 confidence, $K = -0.84$

$\Delta M(0.8) \simeq 98.$

Since $M = \frac{\Gamma}{P} = 2000$, the increase is roughly 5%, from 2000 to 2098.

For 0.9 confidence, $K = -1.28$; for 0.95, $K = -1.65$. We can thus compute

$\Delta M(0.9) \simeq 153$ $\Delta M(0.95) = 201.$

For 4.1b we again have $\Gamma = 250$, but $p = 3/4$, $\mu = 3$, $P = (1-p) = 1/64$. Thus

$$\Delta M = \frac{63}{2}\left[K^2 + \sqrt{K^4 + \frac{998\cdot 64}{63}K^2}\right] - 32$$

$$= 31.5\left[K^2 + \sqrt{K^4 + 1014K^2}\right] - 32 .$$

In this case the nominal M is 64,000 and we have $\Delta M(0.8) = 833$, $\Delta M(0.9) = 1305$, $\Delta M(0.95) = 1711$.